

Shearfree perfect fluids with solenoidal magnetic curvature and a γ -law equation of state

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Abstract. We show that shearfree perfect fluids obeying an equation of state $p = (\gamma - 1)\mu$ are non-rotating or non-expanding under the assumption that the spatial divergence of the magnetic part of the Weyl tensor is zero.

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1. Introduction

The shear-free fluid conjecture claims that for any general relativistic perfect fluid, in which the energy density μ and the pressure p satisfy a barotropic equation of state $p = p(\mu)$ with $p + \mu \neq 0$ [‡], necessarily the expansion θ or the vorticity ω vanishes. The conjecture has been demonstrated in a number of particular cases: constant p (‘dust’ with a cosmological constant) [10, 9, 16], spatial homogeneity [1, 11], $dp/d\mu = 1/3$ [20], $dp/d\mu = -1/3$ or $1/9$ [8, 21], aligned vorticity and acceleration [22], vanishing magnetic part of the Weyl tensor $\mathbf{H} = 0$ [7], $\theta = \theta(\mu)$ [12], $\theta = \theta(\omega)$ [18], Petrov types N [3] and III [4], models in which there is a conformal Killing vector parallel to the fluid flow [6], purely magnetic perfect fluids [8]. It should be noted that this conjecture is not true in Newtonian theory and hence if true in general relativity, then it would highlight essential differences, like that of a well-defined universal time, between the two theories [17]. Although in most of the above mentioned results a tetrad formalism was used, recently it has been shown that a covariant approach can also be successfully employed with considerable potential for future analysis [17, 18, 5].

Motivated by the continued absence of a general proof of the conjecture (or lack of counter-example), we have decided, as an important step forward, to investigate a direct generalisation of Collins’ 1984 result that $\omega\theta = 0$ holds when the magnetic part \mathbf{H} of the Weyl curvature (with respect to the fluid congruence) vanishes. In our analysis,

[‡] otherwise the fluid velocity is not uniquely determined by the geometry and shearfree, rotating and expanding examples are known [15]

we shall make the weaker assumption that \mathbf{H} is solenoidal, that is, that its spatial divergence (see below) vanishes. Interestingly, it has been recently shown [19, 2] that the assumption of third order restrictions, such as $\text{div}\mathbf{H} = 0$ and/or $\text{div}\mathbf{E} = 0$, leads to physically interesting families of perfect fluid solutions. Also, in a classification attempt of these fluids the shearfree sub-family would appear to be a natural first candidate for further investigation. Although a large part of our analysis holds for a general barotropic equation of state $p = p(\mu)$ (such as the positive definiteness of the matrix and the ensuing existence of a Killing vector parallel to the vorticity), at a certain point, we shall need to simplify matters by assuming a γ -law equation of state (with a possible non-vanishing cosmological term "effectively" present by allowing $p = (\gamma - 1)\mu + \text{constant}$). In the general barotropic case with $D^b H_{ab} = 0$, technical difficulties of a very different nature are present and a number of extra subcases remain to be investigated (work in progress).

2. Tetrad choice and relevant equations

We consider shear-free perfect fluid solutions of the Einstein field equations

$$R_{ab} - \frac{1}{2}Rg_{ab} = \mu u_a u_b + p h_{ab}, \quad (1)$$

where \mathbf{u} is the future-pointing (timelike) unit tangent vector to the flow, μ and p are the energy density and pressure of the fluid and $h_{ab} = g_{ab} + u_a u_b$ is the projection tensor into the rest space of the observers with 4-velocity \mathbf{u} . The vanishing of the shear can be expressed by

$$u_{a;b} = \frac{1}{3}\theta h_{ab} + \omega_{ab} - \dot{u}_a u_b \quad (2)$$

where θ is the (rate of) volume expansion, $\dot{\mathbf{u}}$ the acceleration and $\boldsymbol{\omega}$ is the vorticity.

Throughout, we will assume familiarity with the notations and conventions of the orthonormal tetrad formalism of [14]. Crucial in the successful investigation of any problem with tetrad formalisms, is the choice of the tetrad alignment as it can dramatically alter the appearance and complexity of the resulting complete set of equations. We have chosen our tetrad as follows. We begin by aligning \mathbf{e}_0 and \mathbf{e}_3 with \mathbf{u} and $\boldsymbol{\omega}$, respectively, such that $\boldsymbol{\omega} = \omega \mathbf{e}_3 \neq 0$. The relevant variables then become $\mu, p, \omega, \theta, \dot{u}_\alpha$ and the quantities Ω_α , (which determine the rotation of the triad \mathbf{e}_α with respect to a set of Fermi propagated axes), together with the quantities $n_{\alpha\beta}$ and a_α . Latin indices will be spacetime indices while Greek indices will take the values 1, 2 and 3. Further, it is advantageous to replace $n_{\alpha\beta}$ and a_α with the new variables q_α, r_α and n_α , defined by

$$n_{\alpha+1 \ \alpha-1} = (r_\alpha + q_\alpha)/2, \quad a_\alpha = (r_\alpha - q_\alpha)/2, \quad n_{\alpha\alpha} = n_{\alpha+1} + n_{\alpha-1}, \quad (3)$$

expressions which have to be read modulo 3 (for example $\alpha = 3$ gives $n_{12} = (q_3 + r_3)/2$.) The relation between these quantities and the Ricci rotation coefficients can be deduced from the commutators (65). We will also use, as extra (extension) variables, the components of the spatial gradient of the expansion

$$z_\alpha = \partial_\alpha \theta \quad (4)$$

and the (3+1) covariant divergence of the acceleration,

$$j \equiv \dot{u}^a_{;a} = \partial_\alpha \dot{u}^\alpha + \dot{u}^\alpha \dot{u}_\alpha - \dot{u}^\alpha (r_\alpha - q_\alpha). \quad (5)$$

As neither the Jacobi identities nor the field equations contain expressions for the evolution of the Ω_α , it is good practice to choose the triad \mathbf{e}_α such that $\boldsymbol{\Omega} + \boldsymbol{\omega} = 0$. The fact that this is always possible follows from the results of the action of the commutators $[\partial_3, \partial_1]$ and $[\partial_3, \partial_2]$ on p . One finds that $\Omega_1 = \Omega_2 = 0$ after which a rotation in the (12)-plane can be chosen such that $\Omega_3 + \omega = 0$. Herewith, the tetrad is determined up to rotations in the (12)-plane over an angle α satisfying $\partial_0 \alpha = 0$. Noticing that the evolution equations for the quantities $n_{11} - n_{22}$ and n_{12} are identical (see [14]), allows one to further fix the tetrad by making either $n_{11} - n_{22} = 0$ or $n_{12} = 0$. Henceforth our choice will be

$$n_{11} = n_{22} \equiv n. \quad (6)$$

We express the vanishing of the spatial divergence of the magnetic part of the Weyl tensor, $H_{ab} = C_{acbd}u^c u^d$, by means of the Bianchi identity [13],

$$(\text{div} H)_a \equiv D^b H_{ab} = 3\omega^b E_{ab} + (\mu + p)\omega_a. \quad (7)$$

Here the spatial derivative operator is defined by

$$D_a S^{c\dots d}_{e\dots f} = h_a{}^b h^c{}_p \dots h^d{}_q h_e{}^r \dots h_f{}^s \nabla_b S^{p\dots q}_{r\dots s}.$$

This shows that when $D^b H_{ab} = 0$, the vorticity is an eigenvector of the electric part of the Weyl tensor, $E_{ab} = C_{acbd}u^c u^d$, with eigenvalue $-(\mu + p)/3$. With our choice of tetrad this implies

$$E_{13} = E_{23} = 0, \quad (8)$$

$$E_{33} = -\frac{\mu + p}{3}. \quad (9)$$

In order to relate the components of E_{ab} with the spatial gradient of the acceleration, we will make use of the Ricci identity [13]

$$E_{ab} = D_{\langle a} \dot{u}_{b\rangle} - \omega_{\langle a} \omega_{b\rangle} + \dot{u}_{\langle a} \dot{u}_{b\rangle}, \quad (10)$$

where $S_{\langle ab\rangle}$ stands for the spatially projected and trace-free part of S_{ab} .

The basic equations of the formalism are now the Einstein field equations and the Jacobi equations, which we present, using the simplifications above, in the appendix. Henceforth we will assume $\gamma \neq 1$ [10, 16, 9], $\gamma \neq \frac{4}{3}$ [20], $\gamma \neq \frac{2}{3}$ [21], $\gamma \neq \frac{10}{9}$ [21] and, of course $\gamma \neq 0$ (cf. introduction).

We shall argue by contradiction in order to establish that a shearfree fluid under the given conditions satisfies $\omega\theta = 0$. The important step is contained in section 3, where we prove first that, if $\omega\theta \neq 0$, a Killing vector exists parallel to the vorticity. This results in a great simplification of the governing equations and, although this subcase still contains all the intricacies of the ‘general’ problem, it allows us to complete the proof in section 4.

3. Proof of the existence of a Killing vector parallel to ω for $\omega\theta \neq 0$

First notice that the equations (73,76) immediately lead to evolution equations for the variables r_α and q_α ,

$$3\partial_0 r_\alpha = -\theta\dot{u}_\alpha - z_\alpha - \theta r_\alpha \quad (11)$$

$$3\partial_0 q_\alpha = z_\alpha + \theta(\dot{u}_\alpha + q_\alpha) \quad (12)$$

while (66) and the (0α) field equations (78-80) give us the spatial derivatives of ω

$$\partial_1 \omega = \frac{2}{3}z_2 - \omega(q_1 + 2\dot{u}_1) \quad (13)$$

$$\partial_2 \omega = -\frac{2}{3}z_1 + \omega(r_2 - 2\dot{u}_2) \quad (14)$$

$$\partial_3 \omega = \omega(\dot{u}_3 + r_3 - q_3) \quad (15)$$

together with the algebraic restriction

$$n_{33} = \frac{2}{3\omega}z_3. \quad (16)$$

The evolution equation for n follows from (74):

$$\partial_0 n = -\frac{\theta}{3}n. \quad (17)$$

Acting with the commutators $[\partial_0, \partial_\alpha]$ and $[\partial_1, \partial_2]$ on the pressure and using (72) together with the conservation laws

$$\partial_0 \mu = -\theta(\mu + p) \quad (18)$$

$$\partial_\alpha p = -(\mu + p)\dot{u}_\alpha \quad (19)$$

leads to a first set of evolution equations for the acceleration and vorticity:

$$\partial_0 \dot{u}_\alpha = (\gamma - 1)z_\alpha + (\gamma - \frac{4}{3})\theta\dot{u}_\alpha, \quad (20)$$

$$\partial_0 \omega = (\gamma - \frac{5}{3})\omega\theta.$$

The spatial derivatives of the acceleration can be obtained from (10), using (5):

$$\partial_1 \dot{u}_1 = -\frac{1}{3}\omega^2 + \frac{1}{3}j - \dot{u}_2 q_2 + \dot{u}_3 r_3 - \dot{u}_1^2 + E_{11} \quad (21)$$

$$\partial_2 \dot{u}_2 = -\frac{1}{3}\omega^2 + \frac{1}{3}j - \dot{u}_3 q_3 + \dot{u}_1 r_1 - \dot{u}_2^2 + E_{22} \quad (22)$$

$$\partial_3 \dot{u}_3 = -\frac{1}{3}\omega^2 + \frac{1}{3}j - \dot{u}_1 q_1 + \dot{u}_2 r_2 - \dot{u}_3^2 + E_{33} \quad (23)$$

$$\partial_1 \dot{u}_2 = \omega\theta(1 - \gamma) + q_2 \dot{u}_1 + \frac{1}{2}n_{33}\dot{u}_3 - \dot{u}_1 \dot{u}_2 + E_{12} \quad (24)$$

$$\partial_2 \dot{u}_1 = -\omega\theta(1 - \gamma) - r_1 \dot{u}_2 - \frac{1}{2}n_{33}\dot{u}_3 - \dot{u}_1 \dot{u}_2 + E_{12} \quad (25)$$

$$\partial_1 \dot{u}_3 = -r_3 \dot{u}_1 - \frac{1}{2}\dot{u}_2 n_{33} - \dot{u}_1 \dot{u}_3 + E_{13} \quad (26)$$

$$\partial_2 \dot{u}_3 = \frac{1}{2}\dot{u}_1 n_{33} + q_3 \dot{u}_2 - \dot{u}_2 \dot{u}_3 + E_{23} \quad (27)$$

$$\partial_3 \dot{u}_1 = -\frac{1}{2} \dot{u}_2 n_{33} + n \dot{u}_2 + q_1 \dot{u}_3 - \dot{u}_1 \dot{u}_3 + E_{13} \quad (28)$$

$$\partial_3 \dot{u}_2 = \frac{1}{2} \dot{u}_1 n_{33} - n \dot{u}_1 - r_2 \dot{u}_3 - \dot{u}_2 \dot{u}_3 + E_{23} \quad (29)$$

Next we act with the $[\partial_0, \partial_\alpha]$ commutators on ω and θ and use the propagation of (16) along \mathbf{u} in order to obtain expressions for the evolution of z_α and the spatial gradient of j :

$$\begin{aligned} \partial_0 z_1 &= \theta(\gamma - 2)z_1 - \frac{9\gamma - 10}{2}\omega(z_2 + \theta \dot{u}_2) \\ \partial_0 z_2 &= \theta(\gamma - 2)z_2 + \frac{9\gamma - 10}{2}\omega(z_1 + \theta \dot{u}_1) \\ \partial_0 z_3 &= \theta(\gamma - 2)z_3 \end{aligned} \quad (30)$$

and

$$\begin{aligned} \partial_1 j &= \theta z_1(\gamma - 1) - z_2 \frac{-14 + 27\gamma}{6}\omega - \dot{u}_1 j - \frac{-1 + 2\gamma}{2(\gamma - 1)}\dot{u}_1 \mu - \frac{\gamma - 1}{2}\dot{u}_1 p \\ &\quad + \frac{1}{3}\dot{u}_1(18\omega^2 + \theta^2) - \dot{u}_2 \frac{9\gamma - 10}{2}\theta\omega + 4\omega^2 q_1 \end{aligned} \quad (31)$$

$$\begin{aligned} \partial_2 j &= \theta z_2(\gamma - 1) + z_1 \frac{-14 + 27\gamma}{6}\omega - \dot{u}_2 j - \frac{-1 + 2\gamma}{2(\gamma - 1)}\dot{u}_2 \mu - \frac{\gamma - 1}{2}\dot{u}_2 p \\ &\quad + \frac{1}{3}\dot{u}_2(18\omega^2 + \theta^2) + \dot{u}_1 \frac{9\gamma - 10}{2}\theta\omega - 4\omega^2 r_2 \end{aligned} \quad (32)$$

$$\begin{aligned} \partial_3 j &= \theta z_3(\gamma - 1) - \dot{u}_3 j - \frac{-1 + 2\gamma}{2(\gamma - 1)}\dot{u}_3 \mu - \frac{\gamma - 1}{2}\dot{u}_3 p + \frac{1}{3}\dot{u}_3(-18\omega^2 + \theta^2) \\ &\quad - 4(r_3 - q_3)\omega^2. \end{aligned} \quad (33)$$

Now we may evaluate $\sum_\alpha [\partial_0, \partial_\alpha] \dot{u}_\alpha$ by using (5) and (21-23) and this leads to the evolution of j as

$$\begin{aligned} \partial_0 j &= (\gamma - 1)\partial_\alpha z^\alpha + \theta(\gamma - \frac{5}{3})j + (2\gamma - \frac{5}{3})\theta \dot{u}_\alpha \dot{u}^\alpha + (4\gamma - \frac{11}{3})\dot{u}_\alpha z^\alpha \\ &\quad + (\gamma - 1)(q_\alpha - r_\alpha)z^\alpha \end{aligned} \quad (34)$$

Next a long calculation, involving the propagation of the field equations (81-86) along \mathbf{u} and the use of the $[\partial_0, \partial_\alpha]$ commutators on $r_\beta, q_\beta, \dot{u}_\beta$ together with the $[\partial_\alpha, \partial_\beta]$ commutators on θ and ω , allows one to obtain algebraic expressions for the directional derivatives $\partial_\alpha z_\beta$, as well as evolution equations for j and $E_{\alpha\beta}$ (the latter can equally well be obtained by writing out the ‘dot E’ Bianchi identities [13]). The resulting expressions can be simplified by the introduction of the following non-linear combinations of kinematic quantities, each having a clear geometric meaning,

$$U = \dot{u}_1^2 + \dot{u}_2^2, V = \dot{u}_1 z_1 + \dot{u}_2 z_2, W = \dot{u}_1 z_2 - \dot{u}_2 z_1, Z = z_1^2 + z_2^2. \quad (35)$$

We obtain:

a) the evolution of j :

$$\partial_0 j = \frac{6\gamma - 5}{3}(V + \theta U + z_3 \dot{u}_3 + \theta \dot{u}_3^2) + (\gamma - 1)(9\gamma - 10)\omega^2 \theta + \frac{3\gamma - 5}{3}j\theta; \quad (36)$$

b) the evolution of $E_{\alpha\beta}$:

$$\begin{aligned}\partial_0 E_{11} = & -\frac{2}{3}\theta E_{11} \\ & + \frac{2(3\gamma-4)}{9(3\gamma-2)}(2\dot{u}_1 z_1 - \dot{u}_2 z_2 - \dot{u}_3 z_3 + (\omega^2(9\gamma-8) + 2\dot{u}_1^2 - \dot{u}_2^2 - \dot{u}_3^2)\theta),\end{aligned}\quad (37)$$

$$\begin{aligned}\partial_0 E_{22} = & -\frac{2}{3}\theta E_{22} \\ & - \frac{2(3\gamma-4)}{9(3\gamma-2)}(\dot{u}_1 z_1 - 2\dot{u}_2 z_2 + \dot{u}_3 z_3 - (\omega^2(9\gamma-8) - \dot{u}_1^2 + 2\dot{u}_2^2 - \dot{u}_3^2)\theta),\end{aligned}\quad (38)$$

$$\partial_0 E_{12} = -\frac{2}{3}\theta E_{12} + \frac{3\gamma-4}{3(3\gamma-2)}(\dot{u}_2 z_1 + \dot{u}_1 z_2 + 2\dot{u}_1 \dot{u}_2 \theta),\quad (39)$$

$$\partial_0 E_{13} = -\frac{2}{3}\theta E_{13} + \frac{3\gamma-4}{3(3\gamma-2)}(\dot{u}_3 z_1 + \dot{u}_1 z_3 + 2\dot{u}_1 \dot{u}_3 \theta),\quad (40)$$

$$\partial_0 E_{23} = -\frac{2}{3}\theta E_{23} + \frac{3\gamma-4}{3(3\gamma-2)}(\dot{u}_3 z_2 + \dot{u}_2 z_3 + 2\dot{u}_2 \dot{u}_3 \theta);\quad (41)$$

c) the spatial derivatives of z_α :

$$\begin{aligned}\partial_1 z_1 = & -\frac{2(15\gamma-14)}{3(3\gamma-2)}\dot{u}_1 z_1 + \frac{\dot{u}_2(6\gamma-8) + q_2(6-9\gamma)}{3(3\gamma-2)}z_2 + \frac{\dot{u}_3(6\gamma-8) + r_3(9\gamma-6)}{3(3\gamma-2)}z_3 \\ & + \frac{\theta}{3(3\gamma-2)}(\omega^2(2-33\gamma+27\gamma^2) - (6\gamma-8)(2\dot{u}_1^2 - \dot{u}_2^2 - \dot{u}_3^2) + (6-9\gamma)E_{11}),\end{aligned}\quad (42)$$

$$\begin{aligned}\partial_2 z_2 = & -\frac{2(15\gamma-14)}{3(3\gamma-2)}\dot{u}_2 z_2 + \frac{\dot{u}_1(6\gamma-8) + r_1(9\gamma-6)}{3(3\gamma-2)}z_1 + \frac{\dot{u}_3(6\gamma-8) - q_3(9\gamma-6)}{3(3\gamma-2)}z_3 \\ & + \frac{\theta}{3(3\gamma-2)}(\omega^2(2-33\gamma+27\gamma^2) + (6\gamma-8)(\dot{u}_1^2 - 2\dot{u}_2^2 + \dot{u}_3^2) + (6-9\gamma)E_{22}),\end{aligned}\quad (43)$$

$$\begin{aligned}\partial_3 z_3 = & -\frac{2(15\gamma-14)}{3(3\gamma-2)}\dot{u}_3 z_3 + \frac{\dot{u}_1(6\gamma-8) + q_1(6-9\gamma)}{3(3\gamma-2)}z_1 + \frac{\dot{u}_2(6\gamma-8) - r_2(6-9\gamma)}{3(3\gamma-2)}z_2 \\ & + \frac{\theta}{3(3\gamma-2)}(\omega^2(56-78\gamma+27\gamma^2) + (6\gamma-8)U + (16-12\gamma)\dot{u}_3^2 + (6-9\gamma)E_{33}),\end{aligned}\quad (44)$$

$$\begin{aligned}\partial_1 z_2 = & q_2 z_1 - \frac{6(\gamma-1)}{3\gamma-2}(\dot{u}_2 z_1 + \dot{u}_1 z_2) + \frac{1}{2}n_{33}z_3 - \frac{2(3\gamma-4)}{3\gamma-2}\dot{u}_1 \dot{u}_2 \theta - \theta E_{12} \\ & + \omega\left(\frac{\mu+3p}{2} - j + \frac{1}{3}\theta^2 - 2\omega^2\right),\end{aligned}\quad (45)$$

$$\begin{aligned}\partial_2 z_1 = & -r_1 z_2 - \frac{6(\gamma-1)}{3\gamma-2}(\dot{u}_2 z_1 + \dot{u}_1 z_2) - \frac{1}{2}n_{33}z_3 - \frac{2(3\gamma-4)}{3\gamma-2}\dot{u}_1 \dot{u}_2 \theta - \theta E_{12} \\ & - \omega\left(\frac{\mu+3p}{2} - j + \frac{1}{3}\theta^2 - 2\omega^2\right),\end{aligned}\quad (46)$$

$$\partial_3 z_1 = q_1 z_3 - \frac{1}{2}(n_{33} - 2n)z_2 - \frac{6(\gamma-1)}{3\gamma-2}(\dot{u}_3 z_1 + \dot{u}_1 z_3) - \frac{2(3\gamma-4)}{3\gamma-2}\dot{u}_1 \dot{u}_3 \theta - \theta E_{13},\quad (47)$$

$$\partial_3 z_2 = -r_2 z_3 + \frac{1}{2}(n_{33} - 2n)z_1 - \frac{6(\gamma-1)}{3\gamma-2}(\dot{u}_3 z_2 + \dot{u}_2 z_3) - \frac{2(3\gamma-4)}{3\gamma-2}\dot{u}_2 \dot{u}_3 \theta - \theta E_{23},\quad (48)$$

$$\partial_1 z_3 = -r_3 z_1 - \frac{1}{2}n_{33}z_2 - \frac{6(\gamma-1)}{3\gamma-2}(\dot{u}_3 z_1 + \dot{u}_1 z_3) - \frac{2(3\gamma-4)}{3\gamma-2}\dot{u}_1 \dot{u}_3 \theta - \theta E_{13},\quad (49)$$

$$\partial_2 z_3 = q_3 z_2 + \frac{1}{2} n_{33} z_1 - \frac{6(\gamma - 1)}{3\gamma - 2} (\dot{u}_3 z_2 + \dot{u}_2 z_3) - \frac{2(3\gamma - 4)}{3\gamma - 2} \dot{u}_2 \dot{u}_3 \theta - \theta E_{23}. \quad (50)$$

The evolution of the new variables U , V , W and Z is given by

$$\begin{aligned} \partial_0 U &= \frac{2(3\gamma - 4)}{3} \theta U + 2(\gamma - 1)V \\ \partial_0 V &= \frac{2(3\gamma - 5)}{3} \theta V - \frac{9\gamma - 10}{2} \omega W + (\gamma - 1)Z \\ \partial_0 W &= \frac{9\gamma - 10}{2} \theta \omega U + \frac{9\gamma - 10}{2} \omega V + \frac{2(3\gamma - 5)}{3} \theta W \\ \partial_0 Z &= \theta \omega (9\gamma - 10)W + 2(\gamma - 2)\theta Z \end{aligned} \quad (51)$$

The equations given so far describe the behaviour of a general shearfree perfect fluid with a γ -law equation of state (and hence can be useful in further investigations of the $\omega\theta = 0$ conjecture). It is only at this stage that we introduce the condition that the magnetic part of the Weyl tensor is divergence-free. Imposing (8, 9) we deduce from (40, 41)

$$\left(\gamma - \frac{4}{3}\right)(z_3 + 2\theta\dot{u}_3)\dot{u}_1 + \dot{u}_3 z_1 = 0 \quad (52)$$

$$\left(\gamma - \frac{4}{3}\right)(z_3 + 2\theta\dot{u}_3)\dot{u}_2 + \dot{u}_3 z_2 = 0, \quad (53)$$

where the factor $\gamma - \frac{4}{3}$ can be divided out ($\gamma = \frac{4}{3}$ corresponds to a pure radiation perfect fluid, in which case the conjecture $\omega\theta = 0$ is known to hold[20]).

3.1. $z_3 + 2\theta\dot{u}_3 \neq 0$

First suppose that $z_3 + 2\theta\dot{u}_3 \neq 0$. Now propagating equations (52,53) along \mathbf{u} results in a linear system in z_1, z_2 ,

$$\alpha z_1 + \beta z_2 = 0, \quad -\beta z_1 + \alpha z_2 = 0 \quad (54)$$

with

$$\begin{aligned} \alpha &= (\gamma - 1)(z_3 + \dot{u}_3 \theta)^2 + \left(\frac{\mu + 3p}{2} - j + \left(\gamma - \frac{4}{3}\right)\theta^2 - 2\omega^2 \right) \dot{u}_3^2 \\ \beta &= -\frac{9\gamma - 10}{4} (z_3 + \theta\dot{u}_3) \omega \dot{u}_3. \end{aligned} \quad (55)$$

If $\beta \neq 0$ the system (54) has only the 0-solution, and consequently (52,53) would imply $\dot{u}_1 = \dot{u}_2 = 0$. In this case, the acceleration would be parallel to the vorticity and the conjecture $\omega\theta = 0$ follows by [22]. When $\beta = 0$ we have the following possibilities:

- $\gamma = \frac{10}{9}$: in this case the conjecture $\omega\theta = 0$ has been demonstrated [21]
- $\dot{u}_3 = 0$: (52) now implies $\dot{u}_1 = \dot{u}_2 = 0$ which takes us back to the ‘dust’ cases with cosmological constant [9, 10, 16]
- $z_3 + \theta\dot{u}_3 = 0$ and hence, by (52), $\dot{u}_3(z_1 + \theta\dot{u}_1) = \dot{u}_3(z_2 + \theta\dot{u}_2) = 0$. Assuming $\dot{u}_3 \neq 0$ this implies that the spatial gradient of $\log \theta - \int \frac{dp}{\mu+p}$ vanishes, and the fluid flow is irrotational (unless of course $\log \theta - \int \frac{dp}{\mu+p} = 0$, in which case the $\omega\theta = 0$ conjecture would follow from [12]), which contradicts our assumption.

3.2. $z_3 + 2\theta\dot{u}_3 = 0$

If $z_3 + 2\theta\dot{u}_3 = 0$ then (52) implies that $\dot{u}_3 z_1 = \dot{u}_3 z_2 = 0$. If $\dot{u}_3 \neq 0$ propagation of the conditions $z_1 = z_2 = 0$ along \mathbf{u} gives, using (30),

$$\dot{u}_1(9\gamma - 10)\omega\theta = \dot{u}_2(9\gamma - 10)\omega\theta = 0 \quad (56)$$

which again would imply that the vorticity and acceleration are parallel. We therefore conclude that $\dot{u}_3 = 0$ and hence also $z_3 = 0$. By (80) one also has then $n_{33} = 0$. Furthermore, with (26,27) and (49,50) the conditions $\dot{u}_3 = z_3 = 0$ yield

$$r_3 z_1 = q_3 z_2 = 0, \quad r_3 \dot{u}_1 = q_3 \dot{u}_2 = 0, \quad (57)$$

such that, unless acceleration and vorticity are parallel, at least one of q_3 or $r_3 = 0$. Without loss of generality we can suppose $q_3 = 0$. If $r_3 \neq 0$ then $z_1 = \dot{u}_1 = 0$, after which (30) would imply $(9\gamma - 10)\omega(z_2 + \theta\dot{u}_2) = 0$. The spatial gradients of θ and p would then be parallel (with the coefficient of proportionality depending on θ only), which again would imply that the flow is irrotational. We conclude that both $q_3 = r_3 = 0$. A straightforward calculation shows then that there is a Killing vector along \mathbf{e}_3 , although in the sequel this property will not be explicitly used.

4. Proof of $\omega\theta = 0$

From (23,44) and $\dot{u}_3 = z_3 = 0$ one obtains

$$j + 2\omega^2 - 3\dot{u}_1 q_1 + 3\dot{u}_2 r_2 + 3E_{33} = 0 \quad (58)$$

$$(3\gamma - 2)(-3q_1 z_1 + 3r_2 z_2 - 3\theta E_{33}) + (3\gamma - 4)(2\theta U + 2V + (9\gamma - 14)\omega^2\theta) = 0 \quad (59)$$

Propagating (9) along \mathbf{u} gives a third algebraic relation among the same variables, which allows us to express q_1 and r_2 as

$$q_1 = ((\gamma - 1)(p + \mu) + 3(3\gamma - 4)\omega^2) \theta \dot{u}_2 W^{-1} + (j - p - \mu + 2\omega^2) z_2 (3W)^{-1} \quad (60)$$

$$r_2 = ((\gamma - 1)(p + \mu) + 3(3\gamma - 4)\omega^2) \theta \dot{u}_1 W^{-1} + (j - p - \mu + 2\omega^2) z_1 (3W)^{-1} \quad (61)$$

and by which we can rewrite (59) as

$$(3\gamma - 2)^2(p + \mu)\theta + (3\gamma - 4)(4(9\gamma - 8)\theta\omega^2 + 2V + 2\theta U) = 0. \quad (62)$$

Note that $W \neq 0$ since propagating $W = 0$ along \mathbf{u} would imply $(9\gamma - 10)U\omega(z_1 + \theta\dot{u}_1) = 0$ and hence $z_1 + \theta\dot{u}_1 = z_2 + \theta\dot{u}_2 = 0$. Applying the same reasoning as in section 3.2 above would then lead to an irrotational flow.

We now focus on equations (31,32). Applying the $[\partial_1, \partial_2]$ commutator on j and using (13, 14, 21, 22, 24, 25,42, 43, 45, 46, 69, 81) we obtain

$$2(4\gamma - 5)W + (7\gamma - 10)\omega j + (3\gamma - 2)(11\gamma - 8)(p + \mu)\omega + (459\gamma^2 - 1048\gamma + 580)\omega^3 = 0. \quad (63)$$

Propagating this relation twice along \mathbf{u} and simplifying the result using (59) and (63) results in a linear system

$$\begin{bmatrix} 6\gamma(21\gamma - 20)(3\gamma - 4) & (3\gamma - 2)(47\gamma^2 - 76\gamma + 30) \\ 4(5 - 3\gamma)(21\gamma - 20)(3\gamma - 4) & (3\gamma - 2)(47\gamma^2 - 76\gamma + 30) \end{bmatrix} \begin{bmatrix} \omega^2 \\ p + \mu \end{bmatrix} = 0, \quad (64)$$

which clearly implies $\omega = 0$ or $p + \mu = 0$.

5. Discussion

We have demonstrated that a shearfree perfect fluid, obeying an equation of state $p = (\gamma-1)\mu$ and satisfying the ‘solenoidal’ condition $\text{div}H = 0$, is either expansion free or vorticity free. Our proof relies heavily on the subcases discussed earlier in the literature for various values of γ . In addition, we have presented an interesting framework which shows promise for successfully mounting an assault on the shearfree conjecture with the assumption of just a gamma law. Presently we are trying to generalise our result to include an arbitrary barotropic equation of state $p = p(\mu)$ with the solenoidal condition. Preliminary work shows that almost all relations in the present work can be generalised (in particular the fact that $\omega\theta \neq 0$ would imply the existence of a Killing vector parallel to the vorticity). Some technical difficulties still remain to be dealt with in the final stage of the proof. This is primarily due to the resulting more complicated nature of equation (63) and its subsequent derivatives. A proof of the conjecture would be very desirable, as the classification of the shearfree case would form a necessary and natural first step in the study of purely ‘solenoidal’ perfect fluids.

6. Appendix

Commutator relations, using $\sigma_{\alpha\beta} = 0$:

$$\begin{aligned}
[\partial_0, \partial_1] &= \dot{u}_1\partial_0 - \theta_1\partial_1 + (\omega_3 + \Omega_3)\partial_2 - (\omega_2 + \Omega_2)\partial_3 \\
[\partial_0, \partial_2] &= \dot{u}_2\partial_0 - (\omega_3 + \Omega_3)\partial_1 - \theta_2\partial_2 + (\omega_1 + \Omega_1)\partial_3 \\
[\partial_0, \partial_3] &= \dot{u}_3\partial_0 + (\omega_2 + \Omega_2)\partial_1 - \theta_3\partial_3 - (\omega_1 + \Omega_1)\partial_2 \\
[\partial_1, \partial_2] &= -2\omega_3\partial_0 + q_2\partial_1 + r_1\partial_2 + n_{33}\partial_3 \\
[\partial_2, \partial_3] &= -2\omega_1\partial_0 + q_3\partial_2 + r_2\partial_3 + n_{11}\partial_1 \\
[\partial_3, \partial_1] &= -2\omega_2\partial_0 + r_3\partial_1 + q_1\partial_3 + n_{22}\partial_2
\end{aligned} \tag{65}$$

Using in addition the simplifications $\omega_1 = \omega_2 = \Omega_1 = \Omega_2 = 0$, $\Omega_3 = -\omega_3$, $n_{11} = n_{22} = n$ one obtains the:

Jacobi equations:

$$\partial_3\omega = \omega(\dot{u}_3 + r_3 - q_3) \tag{66}$$

$$\partial_1n + \partial_2r_3 + \partial_3q_2 - n(r_1 - q_1) - r_3r_2 + q_3q_2 = 0 \tag{67}$$

$$\partial_2n + \partial_3r_1 + \partial_1q_3 - n(r_2 - nq_2) - r_3r_1 + q_3q_1 = 0 \tag{68}$$

$$\partial_1r_2 + \partial_2q_1 + \partial_3n_{33} - \frac{2}{3}\theta\omega - r_2r_1 + q_2q_1 - n_{33}(r_3 - q_3) = 0 \tag{69}$$

$$\partial_2\dot{u}_3 - \partial_3\dot{u}_2 - n\dot{u}_1 - q_3\dot{u}_2 - r_2\dot{u}_3 = 0 \tag{70}$$

$$\partial_3\dot{u}_1 - \partial_1\dot{u}_3 - n\dot{u}_2 - r_3\dot{u}_1 - q_1\dot{u}_3 = 0 \tag{71}$$

$$2\partial_0\omega + \partial_1\dot{u}_2 - \partial_2\dot{u}_1 - \dot{u}_1q_2 - \dot{u}_2r_1 - \dot{u}_3n_{33} + \frac{4}{3}\theta\omega = 0 \tag{72}$$

$$\partial_0(r_\alpha - q_\alpha) - \frac{1}{3}\partial_\alpha\theta + z_\alpha + \frac{\theta}{3}(r_\alpha - q_\alpha + 2\dot{u}_\alpha) = 0 \quad (73)$$

$$3\partial_0 n + 3\partial_3\omega + n\theta - 3\omega(\dot{u}_3 + r_3 - q_3) = 0 \quad (74)$$

$$3\partial_0 n_{33} + 3\partial_3\omega + n_{33}\theta - 3\omega(\dot{u}_3 + r_3 - q_3) = 0 \quad (75)$$

$$\partial_0(r_\alpha + q_\alpha) + \frac{1}{3}\theta(r_\alpha + q_\alpha) = 0 \quad (76)$$

Einstein equations:

$$\partial_0\theta = -\frac{1}{3}\theta^2 + 2\omega^2 - \frac{1}{2}(\mu + 3p) + j \quad (77)$$

$$2/3z_1 + \partial_2\omega - \omega(r_2 - 2\dot{u}_2) = 0 \quad (78)$$

$$2/3z_2 - \partial_1\omega - \omega(q_1 + 2\dot{u}_1) = 0 \quad (79)$$

$$2/3z_3 - \omega n_{33} = 0 \quad (80)$$

$$\begin{aligned} -\partial_1 r_2 + \partial_2 q_1 - r_1 r_2 - q_1 q_2 - 2r_2 q_1 - 2r_3 n + n_{33}(r_3 + q_3) - 2q_3 n = \\ -\partial_1 \dot{u}_2 - \partial_2 \dot{u}_1 - 2\dot{u}_1 \dot{u}_2 + \dot{u}_1 q_2 - \dot{u}_2 r_1 \end{aligned} \quad (81)$$

$$\begin{aligned} -\partial_2 r_3 + \partial_3 q_2 + \partial_1 n - \partial_1 n_{33} - n(r_1 - q_1) - 2q_1 n_{33} - r_2 r_3 - q_2 q_3 - 2r_3 q_2 = \\ -\partial_2 \dot{u}_3 - \partial_3 \dot{u}_2 + \dot{u}_1(n_{33} - n) - 2\dot{u}_2 \dot{u}_3 + \dot{u}_2 q_3 - \dot{u}_3 r_2 \end{aligned} \quad (82)$$

$$\begin{aligned} \partial_1 q_3 - \partial_3 r_1 + \partial_2 n_{33} - \partial_2 n - r_2(2n_{33} - n) - nq_2 - r_1 r_3 - q_1 q_3 - 2r_1 q_3 = \\ -\partial_1 \dot{u}_3 - \partial_3 \dot{u}_1 - \dot{u}_1(2\dot{u}_3 + r_3) + \dot{u}_2(n - n_{33}) + \dot{u}_3 q_1 \end{aligned} \quad (83)$$

$$\begin{aligned} -\partial_1 r_1 + \partial_1 q_1 + \partial_2 q_2 - \partial_3 r_3 + q_2^2 + r_3^2 + \frac{1}{2}n_{33}^2 - nn_{33} + r_1^2 + q_1^2 - r_2 q_2 - r_3 q_3 = \\ \frac{2}{9}\theta^2 + \frac{8}{3}\omega^2 - \frac{1}{3}(2\mu - j) - \partial_1 \dot{u}_1 - \dot{u}_1^2 + \dot{u}_3 r_3 - \dot{u}_2 q_2 \end{aligned} \quad (84)$$

$$\begin{aligned} -\partial_2 r_2 + \partial_2 q_2 - \partial_1 r_1 + \partial_3 q_3 + q_3^2 + r_1^2 + \frac{1}{2}n_{33}^2 - nn_{33} + r_2^2 + q_2^2 - r_1 q_1 - r_3 q_3 = \\ \frac{2}{9}\theta^2 + \frac{8}{3}\omega^2 - \frac{1}{3}(2\mu - j) - \partial_2 \dot{u}_2 - \dot{u}_2^2 + \dot{u}_1 r_1 - \dot{u}_3 q_3 \end{aligned} \quad (85)$$

$$\begin{aligned} -\partial_3 r_3 + \partial_3 q_3 + \partial_1 q_1 - \partial_2 r_2 + q_1^2 + r_2^2 - \frac{1}{2}n_{33}^2 + r_3^2 + q_3^2 - r_1 q_1 - r_2 q_2 = \\ \frac{2}{9}\theta^2 + \frac{1}{3}(2\omega^2 - 2\mu + j) - \partial_3 \dot{u}_3 - \dot{u}_3^2 + \dot{u}_2 r_2 - \dot{u}_1 q_1 \end{aligned} \quad (86)$$

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